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Domain perturbation method and shape of a bubble in a uniform flow of an inviscid liquid

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Abstract – In many problems with a free boundary there is defined a small parameter, ε, for which the solution is sometimes known for a particular value, $\varepsilon = 0$, and the general solution is obtained as a series in the parameter. To find this solution, the equations can be written on a reference configuration and solved in a fixed domain. The purpose of this study is to show that this method of domain perturbation is a good one. The range of validity of this method will be studied on the model example of the irrotational flow of a perfect fluid around a bubble. The radius of convergence of the series solution will be determined, as will the nature of the solution in the neighbourhood of the first real singularity. © 1999 Éditions scientifiques et médicales Elsevier SAS

domain perturbation method / bubble shape / inviscid liquid

1. Introduction

For certain free boundary problems in fluid mechanics, a parameter is involved that is usually non-dimensional, and which we will refer to as ε. All quantities, i.e. the domain Ω_ε occupied by the fluid and the mechanical quantities such as velocity, pressure, ..., generically denoted by u_ε depend on ε. A typical example is the study of a rotating fluid mass, see Appell [1], with interfacial tension and/or gravitation. In these cases the parameter is the Bond number, which is proportional to the square of the angular velocity. Very often for a particular value of this parameter, say $\varepsilon = 0$, a solution $\Omega_0,\ u_0$, is known. It is natural to seek a general solution as a perturbation series in the parameter. For a free surface problem in general, the domain perturbation method consists in taking as principal unknown the transformation field $\boldsymbol{x} = \boldsymbol{T}(\boldsymbol{X}, \varepsilon)$ from the initial known reference position Ω_0 to the unknown position. In fact, this field depends on ε.

All the equations of the problem must be written in the 'Lagrange' variables corresponding to the reference position, see *figure 1*. This kind of approach has been used before, see Joseph and Fosdick [2], Sattinger [3], Brancher and Séro-Guillaume [4] for example, but its validity, in particular the radius of convergence of the series obtained, has not yet been tested. In this paper, we will study the validity of this kind of method on a generic case: the shape of an incompressible inviscid fluid bubble, in the potential flow of an incompressible inviscid fluid in absence of gravity. See *figure 2* for the notation.

Let $\boldsymbol{v} = \boldsymbol{V}_\infty + \nabla u$ be the velocity of the flow and u the potential due to the presence of the bubble, with $u \to 0$ and $\nabla u \to 0$ at infinity. The equations for u are:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^e, \\ \dfrac{\partial u}{\partial n} = -\boldsymbol{V}_\infty \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases} \tag{1}$$

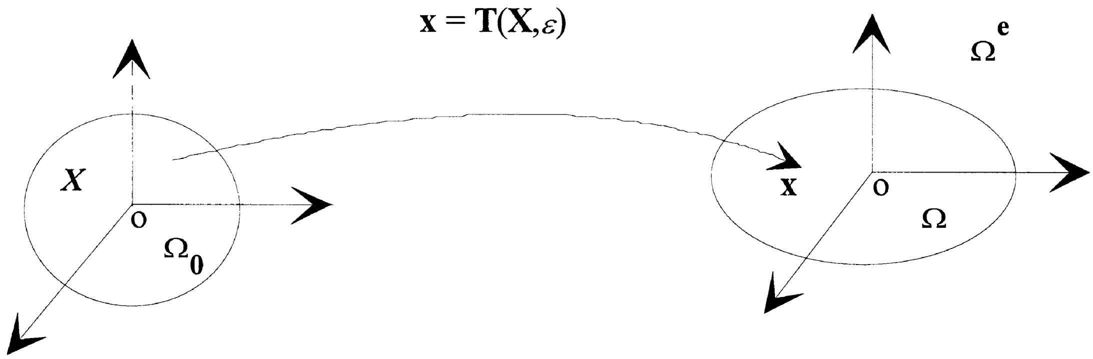

Figure 1. Lagrangian representation of the domain.

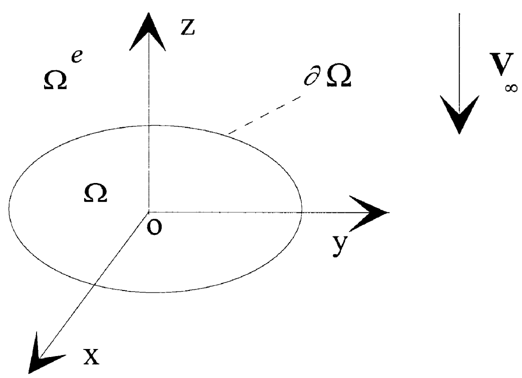

Figure 2. Notation used.

It is assumed that there is no motion inside the bubble, so that the pressure p_0 is constant. The momentum equations reduce to the Bernoulli relation $\rho \boldsymbol{v}^2/2 + p = \rho \boldsymbol{V}_\infty^2/2 + p_\infty$, where p_∞ is the pressure at infinity. In light of the Laplace relation $p = p_0 - \sigma C$, where σ is the surface tension and C the mean curvature of the surface $\partial\Omega$, the Bernoulli relation reduces to:

$$\rho\left[\frac{1}{2}|\nabla u|^2 + \nabla u.\boldsymbol{V}_\infty\right] - \sigma C = p_\infty - p_0. \tag{2}$$

For $\boldsymbol{V}_\infty^2 = 0$ the equilibrium shape is a sphere, so that $\boldsymbol{V}_\infty^2$ could be used as the development parameter ε. Once Eqs (1) and (2) are transported back to the sphere, we get a new system of partial differential equations, say $F(T, X, \varepsilon) = 0$ written on the known domain Ω_0. The set of all transformations $\boldsymbol{x} = \boldsymbol{T}(\boldsymbol{X}, \varepsilon)$ can be considered as a one-parameter group of transformations, and so we can say $\boldsymbol{x} = (\boldsymbol{Id} + \varepsilon \boldsymbol{T}_1 + \varepsilon^2 \boldsymbol{T}_2 + \varepsilon^3 \boldsymbol{T}_3 + \cdots)(\boldsymbol{X})$ where $\boldsymbol{Id}$ is the identity. Using symbolic calculus codes, it is now possible to calculate a sufficient number of terms in the series, to study precisely the solution. It will be shown that one can obtain the radius of convergence of the series, and study its singularities as well as the nature of the bifurcation. In 1974, Schwartz [5] studied the radius of convergence of series of solution for waves on fluids. However the co-ordinates were Eulerian and in the paper [6] he discourages the use of Lagrangian co-ordinates because of the non linearity of the operators.

2. Solution of the perturbation domain problem

2.1. Statement of the problem

Firstly the number of variables are reduced by setting $u=(V_\infty R)u^*,\ r=Rr^*$ with $\boldsymbol{V}_\infty=-V_\infty \boldsymbol{e}_z$. R is the radius of the sphere. Equations (1) and (2) become:

$$\begin{cases} \Delta u^* = 0 & \text{in } \Omega^e, \\ \dfrac{\partial u^*}{\partial n} = \boldsymbol{e}_z.\boldsymbol{n} & \text{on } \partial\Omega, \end{cases} \tag{3}$$

$$\frac{R\rho V_\infty^2}{\sigma}\left[\frac{1}{2}|\nabla u^*|^2 - \nabla u^*.\boldsymbol{e}_z\right] - RC = \frac{R(p_\infty - p_0)}{\sigma}. \tag{4}$$

Let us set $\varepsilon = R\rho V_\infty^2/\sigma,\ RC = C^*,\ k = R(p_\infty - p_0)/\sigma$. ε is the Weber number, k a constant depending on ε. From now on, we will note u^* by u, and C^* by C.

In the spherical co-ordinate system r, θ, ψ, defined in *figure 3*, the form of the transformation field is chosen:

$$\boldsymbol{T} = rg(\theta,\psi,\varepsilon)\boldsymbol{e}_r.$$

With this notation, $r = Rg(\theta,\psi,\varepsilon)$ is the equation of the interface, and $g(\theta,\psi,0)=1$. If $\boldsymbol{V}_\infty$ is changed to $-\boldsymbol{V}_\infty$, the configuration is unchanged, then by symmetry:

$$g(\pi-\theta,\psi,\varepsilon) = g(\theta,\psi,\varepsilon).$$

The transformation must satisfy volume conservation, which can be expressed:

$$\int_0^\pi \int_0^{2\pi} g^3 \sin\theta\, d\theta\, d\psi = \text{Const.} \tag{5}$$

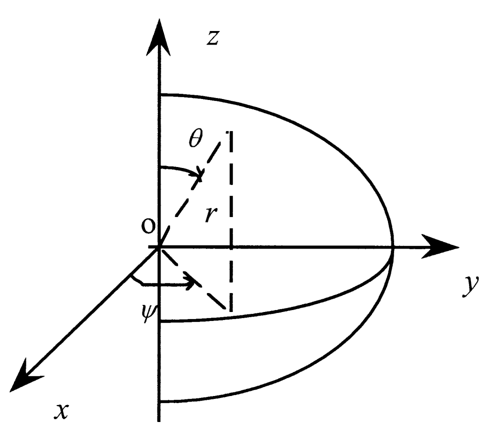

Figure 3. Definition of the spherical co-ordinates.

The equations transported to the reference configuration are:

$$\begin{cases} \nabla \cdot \big(|\det \boldsymbol{T}'|\boldsymbol{T}'^{-1\,t}\boldsymbol{T}'^{-1}\nabla u\big) = 0 & \text{for } r > 1,\\ {}^t\boldsymbol{T}'^{-1}\nabla u \cdot {}^t\boldsymbol{T}'^{-1}\boldsymbol{e}_z = {}^t\boldsymbol{T}'^{-1}\boldsymbol{e}_z \cdot {}^t\boldsymbol{T}'^{-1}\boldsymbol{n} & \text{for } r = 1,\\ \varepsilon\left[\frac{1}{2}\left|{}^t\boldsymbol{T}'^{-1}\nabla u\right|^2 - {}^t\boldsymbol{T}'^{-1}\nabla u \cdot {}^t\boldsymbol{T}'^{-1}\boldsymbol{e}_z\right] - C = k & \text{for } r = 1,\end{cases}$$

where $\boldsymbol{T}'$, $\boldsymbol{T}'^{-1}$, ${}^t\boldsymbol{T}'$ denote, respectively, the jacobian, the inverse and the transpose of the jacobian of $\boldsymbol{T}$.

For the sake of simplicity, we look for an axisymmetrical solution, i.e. one that is independent of the angle ψ. Therefore, the first equation can be written:

$$\begin{aligned}&\left[\left(g + \frac{g'^2}{g}\right)u_r - \frac{g'}{r}u_\theta\right]_r + \frac{2}{r}\left[\left(g + \frac{g'^2}{g}\right)u_r - \frac{g'}{r}u_\theta\right]\\ &\quad + \frac{1}{r}\left[\frac{g}{r}u_\theta - g'u_r\right]_\theta + \frac{\cos\theta}{r\sin\theta}\left[\frac{g}{r}u_\theta - g'u_r\right] = 0 \quad \text{for } r > 1.\end{aligned} \tag{6}$$

The boundary condition is:

$$[g^2 + g'^2]u_r - g'gu_\theta - [g^2 + g'^2]\cos\theta - g'g\sin\theta = 0 \quad \text{for } r = 1 \tag{7}$$

and the Bernoulli relation becomes:

$$\varepsilon\left\{\frac{1}{2}\left[u_r^2 + \left(u_\theta - \frac{g'u_r}{g}\right)^2\right] - \cos\theta u_r + \left(\sin\theta + \frac{g'\cos\theta}{g}\right)\left(u_\theta - \frac{g'u_r}{g}\right)\right\} - \left(C + kg^2\right) = 0, \tag{8}$$

where

$$C = -\left\{\frac{g''g - 2g'^2 - g^2}{(g^2 + g'^2)^{3/2}} + \frac{g'\cos\theta - g\sin\theta}{g\sin\theta(g^2 + g'^2)^{1/2}}\right\}$$

is the mean curvature of the deformed configuration.

2.2. *Existence and uniqueness by the implicit function theorem*

Let $F_1(u,g)$, $F_2(u,g)$, $F_3(u,g,\varepsilon)$ denote the left hand side of relations (6), (7) and (8) respectively, and:

$$F(u,g,\varepsilon) = \left\{F_1(u,g), F_2(u,g), F_3(u,g,\varepsilon)\right\}.$$

Now Eqs (6), (7) and (8) can be written:

$$\begin{aligned} F_1(u,g) &= 0,\\ F_2(u,g) &= 0, \quad \text{or equivalently} \quad F(u,g,\varepsilon) = 0.\\ F_3(u,g,\varepsilon) &= 0,\end{aligned} \tag{9}$$

We will now briefly outline how to obtain the solution by applying the implicit function theorem, not only for mathematical reasons but because the implicit function theorem is the basis of the method for obtaining the solution. Only variations δg of g which preserve the volume will be considered. These are such that:

$$\int_0^{\pi}\int_0^{2\pi} g^2 \delta g \sin\theta \, d\theta \, d\psi = 0. \tag{10}$$

The condition (5) can be ignored, provided the variations of g satisfy relation (10).

Let u_0 be the solution of the system $F_1(u,1)=0,\ F_2(u,1)=0$, i.e. of:

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega_0^e, \\ u_{0r} = \cos\theta & \text{on } \partial\Omega_0 \end{cases} \tag{11}$$

with appropriate conditions at infinity. The implicit function theorem is first applied to the system $F_1(u,g)=0,\ F_2(u,g)=0$ at the point $(u_0,1)$, which is equivalent to solving an exterior Neumann problem for the sphere. Therefore u is defined as a function of g: $u(g)$. If $u(g)$ is substituted in $F_3(u,g,\varepsilon)$ we get an equation in g and ε: $F_4(g,\varepsilon)=F_3(u(g),g,\varepsilon)$. This equation can be solved again by applying the implicit function theorem to $F_4(g,\varepsilon)=0$ at the point $(1,0)$. In order to apply the implicit function theorem it must be proved that $(\partial F_4/\partial g)(1,0)$ is invertible. Where it can be seen from:

$$\frac{\partial F_4}{\partial g}(g,\varepsilon) = \frac{\partial F_3}{\partial g}(u,g,\varepsilon) + \frac{\partial F_3}{\partial u}(u,g,\varepsilon)\cdot\frac{\partial u}{\partial g}(g),$$

that $(\partial F_3/\partial u)(u_0,1,0)=0$. After some algebraic transformations, we arrive at:

$$\frac{\partial F_3}{\partial g}(u_0,1,0)\eta = \eta'' + \frac{\cos\theta}{\sin\theta}\eta' + 2\eta$$

so that $(\partial F_4/\partial g)(1,0)$ is invertible if the differential equation

$$\eta'' + \frac{\cos\theta}{\sin\theta}\eta' + 2\eta = \zeta$$

has a unique solution for any ζ in an appropriate space. This last result is easily obtained.

2.3. Analytical solution

The series solution can now be calculated. As the functionals F_i are analytic in u, g and ε, the solution given by the implicit function theorem is analytic in ε see Sattinger [7]. Therefore the solution has the form:

$$\begin{cases} u = u(r,\theta,\varepsilon) = \sum_{n\geqslant 0}\frac{\varepsilon^n}{n!}v_n(r,\theta), & \text{(12.a)} \\ g = g(\theta,\varepsilon) = \sum_{n\geqslant 0}\frac{\varepsilon^n}{n!}f_n(\theta), & \text{(12.b)} \\ k = \sum_{n\geqslant 0}\frac{\varepsilon^n}{n!}k_n. & \text{(12.c)} \end{cases}$$

The iterative structure for solving such problems is the same in all problems of perturbation, and is described below.

Let us assume that the equation $G(u,\varepsilon)=0$ has to be solved by the implicit function theorem around the point $(u=u_0,\ \varepsilon=0)$. If the asymptotic series solution is denoted by $u=\sum_{n\geqslant 0}(\varepsilon^n/n!)u_n,\ u_n$ is solution of the linear problem:

$$L_0[u_n]=A_n(0,u_0,\ldots,u_{n-1}), \tag{13}$$

where L_0 is the linear operator $(\partial G/\partial u)(u_0,0)$, and can be calculated recursively. The nth derivative $u^{(n)}=\partial^n u/\partial\varepsilon^n$ satisfies the relation:

$$\frac{\partial G}{\partial\varepsilon}\big(u(\varepsilon),\varepsilon\big)\left[\frac{\partial^n u}{\partial\varepsilon^n}\right]=A_n\big[\varepsilon,u^{(0)},u^{(1)},\ldots,u^{(n-1)}\big]. \tag{14}$$

If (14) is differentiated with respect to ε one obtains the recurrence relation:

$$A_{n+1}=\frac{dA_n}{d\varepsilon}-\frac{\partial^2 G}{\partial u\partial\varepsilon}\big(u(\varepsilon),\varepsilon\big)\big[u^{(n)}\big]-\frac{\partial^2 G}{\partial u^2}\big(u(\varepsilon),\varepsilon\big)\big[u^{(n)},u^{(n)}\big]. \tag{15}$$

$d/d\varepsilon$ is the total derivative, that is:

$$\frac{d}{d\varepsilon}=\frac{\partial}{\partial\varepsilon}+\sum_{i>0}u^{(i+1)}\frac{\partial}{\partial u^{(i)}}.$$

Setting $\varepsilon=0$ in (14) we get (13). The successive orders can be calculated with a symbolic calculus code. We have stipulated the iterative structure to emphasize the fact that the left hand side of (13) is the same to all orders, but the number of terms on the right side grows as a factorial. There is a 'combinatorial blow up' for the number of terms with respect to the order, which is why it is usually possible to obtain only a limited number of terms. Twenty terms of the expansion has been obtained for this problem. For the sake of clarity the details of calculations for the three first order terms are given.

Order 1: The equations for the velocity potential are:

$$\begin{cases}\Delta\nu_0=0, & r<1,\\ \nu_{0r}=\cos\theta, & r=1,\\ \nu_0\to 0, & r\to\infty.\end{cases} \tag{16}$$

The Bernoulli relation and the conservation of volume give:

$$k_0=-2,\qquad f_0=1. \tag{17}$$

The solution of (16) is:

$$\nu_0(r,\theta)=-\frac{\cos\theta}{2r^2}. \tag{18}$$

The problem is solved at this order.

Order 2: The Bernoulli relation gives:

$$\frac{1}{2}\nu_{0r}^2-\nu_{0r}\cos\theta+\frac{\nu_{0\theta}^2}{2}+\sin\theta\,\nu_{0\theta}+\left(f_1''+\frac{\cos\theta}{\sin\theta}f_1'+2f_1\right)=k_1.$$

With (18), this relation becomes:

$$f_1'' + \frac{\cos\theta}{\sin\theta} f_1' + 2 f_1 - \frac{9}{8}\cos^2\theta + \frac{5}{8} = k_1. \tag{19}$$

The constant is determined using the conservation of volume, and the solution of (19) is:

$$f_1(\theta) = \frac{3}{32}\big(1 - 3\cos^2\theta\big) = -\frac{3}{16} P_2(\xi), \qquad k_1 = \frac{1}{4},$$

$P_n(x)$ is the nth Legendre polynomial, and $\xi = \cos\theta$.

The equations for the velocity potential are:

$$\begin{cases} \Delta\nu_1 = -f_1\nu_{0rr} + \dfrac{f_1''\nu_{0r}}{r} + \dfrac{\cos\theta}{r\sin\theta} f_1'\nu_{0r} - \dfrac{2}{r} f_1\nu_{0r} - \dfrac{f_1\nu_{0\theta\theta}}{r} + \dfrac{2 f_1\nu_{0\theta r}}{r} - \dfrac{\cos\theta}{\sin\theta} f_1\nu_{0\theta}, & r < 1, \\ \nu_{1r} = \big(f_1'\nu_{0\theta} + f_1''\sin\theta + 2 f_1\cos\theta - 2 f_1\nu_{0r}\big), & r = 1, \\ \nu_1 \to 0, & r \to \infty \end{cases} \tag{20}$$

which can be solved, and the solution is:

$$\nu_1(r,\theta) = -\frac{27}{160 r^2} P_1(\xi) + \frac{9}{8}\left(\frac{1}{8r^4} - \frac{1}{10r^2}\right) P_3(\xi). \tag{21}$$

Order 3: Bernoulli's relation gives:

$$f_2'' + \frac{\cos\theta}{\sin\theta} f_2' + 2 f_2 + \frac{1}{256}\big(531\cos^4\theta - 594\cos^2\theta + 147\big) = k_2. \tag{22}$$

The solution of (22) is:

$$f_2(\theta) = \frac{1}{4480}\big(118 P_4(\xi) - 405 P_2(\xi) - 63\big), \qquad k_2 = 3/16. \tag{23}$$

It can be seen that only the even-order Legendre polynomials are involved in the expression of the shape because of the symmetry of the problem.

Reordering the summation, we can write the expression (12.b) on the basis of Legendre polynomials:

$$g(\theta,\varepsilon) = \sum_{n=0}^{\infty} \beta_n(\varepsilon) P_{2n}(\xi) \quad \text{with } \xi = \cos\theta \tag{24}$$

and

$$\beta_n(\varepsilon) = \sum_{k\geqslant n} \beta_n^k \varepsilon^k. \tag{25}$$

The bubble shape for different values of ε are plotted in *figure 4*.

2.4. Radius of convergence of the series solution

It is necessary to estimate the radius of convergence of the series solution. This will be done using the expressions (24)–(25). In fact, the radius of convergence of each $\beta_n(\varepsilon)$ should be determined, but the number

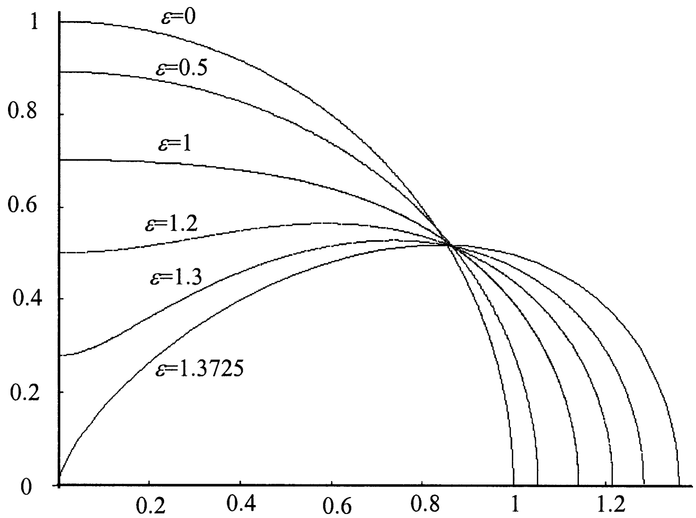

Figure 4. Shapes of equilibrium for different values of ε.

of terms of each series diminishes as n increases, so that the determination of the radius is less and less precise. Only the first ten terms in (24) and the polar and equatorial radius of the shape, i.e. $a(\varepsilon) = g(0, \varepsilon)$ and $b(\varepsilon) = g(\pi/2, \varepsilon)$, have been tested using the Sykes–Domb method [8]. This method is the following.

Let us consider the series:

$$f(\varepsilon) = \sum_{n=0}^{\infty} u_n \varepsilon^n$$

and let us suppose that the function $f(\varepsilon)$ has a singularity of the form:

$$\begin{cases} (\varepsilon - \varepsilon_0)^\gamma & \text{for } \gamma \neq 0, 1, \ldots, n, \ldots, \\ (\varepsilon - \varepsilon_0)^\gamma \log(\varepsilon - \varepsilon_0) & \text{for } \gamma = 0, 1, \ldots, n, \ldots. \end{cases} \tag{26}$$

The D'Alembert ratio $|u_n/u_{n-1}|$ satisfies the relation:

$$\left|\frac{u_n}{u_{n-1}}\right| \approx \pm\frac{1}{\varepsilon_0}\left(1 - \frac{1+\gamma}{n}\right). \tag{27}$$

In any case, even if the singularity is not of the form (26), the function $1/n \to |u_n/u_{n-1}|$ crosses the axis $1/n = 0$ at the inverse of the radius of convergence. But by Fabry's theorem, see Dienes [9], it is known that if the ratio u_n/u_{n-1} converges to a certain positive value, say $1/\varepsilon_0$, then ε_0 is the radius of convergence and is a singularity of the function. Therefore, instead of the D'Alembert ratio, we analyze and plot the ratios without the absolute value. The values of this ratio for $\beta_0(\varepsilon), \beta_1(\varepsilon), a(\varepsilon), b(\varepsilon)$, are given in *table I*. The Sykes–Domb graph for the ratio β_0^{k+1}/β_0^k for $\beta_0(\varepsilon)$ is plotted in *figure 5*.

From *table I* or *figure 5* one might say that the ratio β_0^{k+1}/β_0^k converges. However it is extremely difficult to conclude on a precise value of convergence. Some convergence acceleration process has to be used, see Bender and Orszag [10]. Since the Shanks transformation is not appropriate for this kind of convergence, the Richardson extrapolation process has been used. The Richardson extrapolation for a sequence S_n, is defined as

Table I. D'Alembert ratio for $\beta_0(\varepsilon), \beta_1(\varepsilon), a(\varepsilon), b(\varepsilon)$.

n	$\beta_0(\varepsilon)$	$\beta_1(\varepsilon)$	$a(\varepsilon)$	$b(\varepsilon)$
3	0.4358	0.5616	0.6157	0.5669
4	0.6883	0.5214	0.4213	0.4958
5	0.6157	0.6317	0.5528	0.6234
6	0.6767	0.6471	0.6131	0.6275
7	0.6819	0.6849	0.6494	0.6704
8	0.7029	0.7016	0.6728	0.6837
9	0.7129	0.7182	0.6888	0.7009
10	0.7297	0.7248	0.7004	0.7097
11	0.7352	0.7314	0.7093	0.7167
12	0.7397	0.7367	0.7164	0.7230
13	0.7435	0.7415	0.7222	0.7281
14	0.7468	0.7460	0.7271	0.7325
15	0.7496	0.7494	0.7313	0.7363
16	0.7521	0.7519	0.7350	0.7396
17	0.7543	0.7541	0.7382	0.7425
18	0.7563	0.7561	0.7410	0.7451
19	0.7580	0.7579	0.7436	0.7475
20	0.7595		0.7459	0.7496

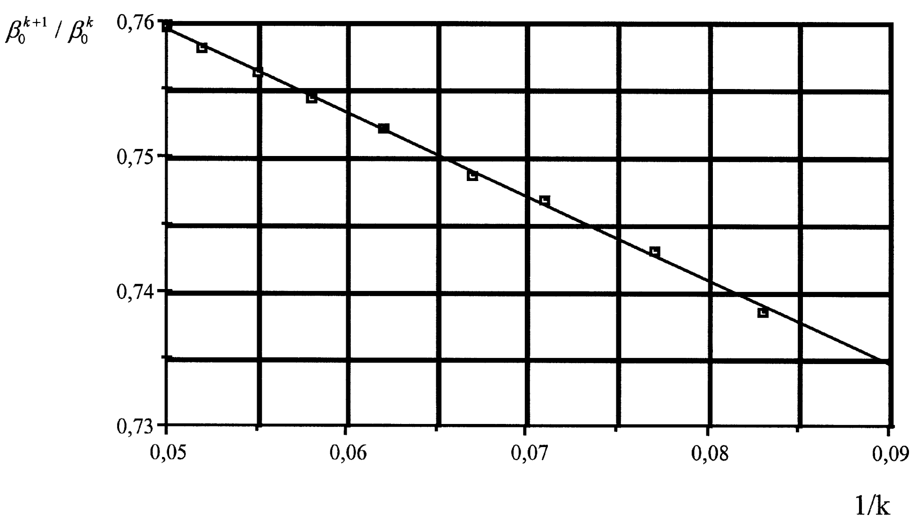

Figure 5. Sykes–Domb graph for $\beta_0(\varepsilon)$.

the sequence:

$$q_{mn} = \sum_{k=0}^{m} \frac{S_{n+k}(n+k)^m(-1)^{k+m}}{k!(m-k)!}.$$

If the sequence S_n has the asymptotic expansion

$$S_n = Q_0 + Q_1 n^{-1} + Q_2 n^{-2} + \cdots, \quad n \to \infty,$$

Table II. Values of the Richardson extrapolation q_{nm} for β_0^{k+1}/β_0^k.

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
10	0.795387	0.795389	0.795395	0.795397	0.795399
11	0.795391	0.795395	0.795400	0.795405	0.795408
12	0.795396	0.795399	0.795405	0.795419	0.795419
13	0.795405	0.795406	0.795419	0.795419	0.795419
14	0.795419	0.795419	0.795419	0.795419	0.795419
15	0.795419	0.795419	0.795419	0.795419	0.795419

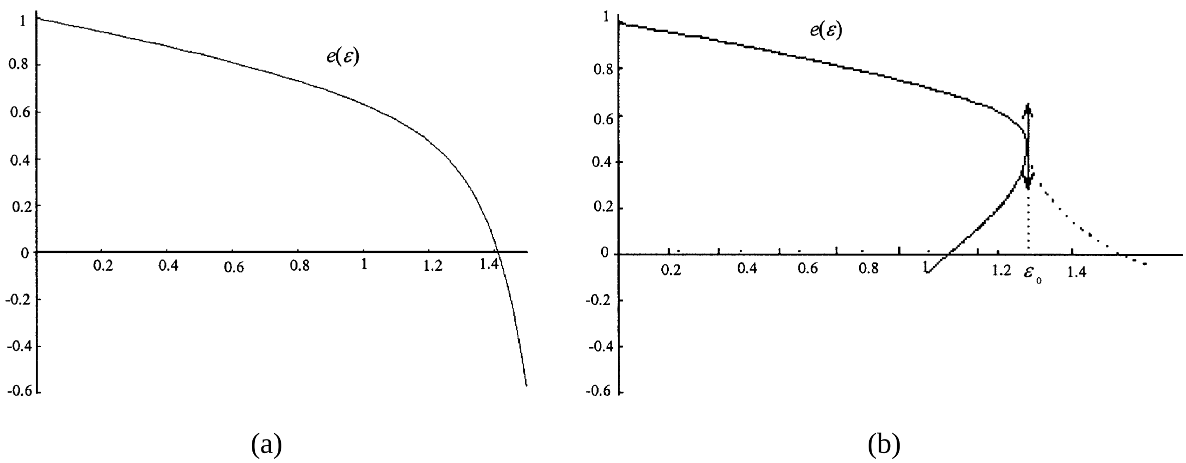

Figure 6. Aspect ratio versus ε; (a) calculated, (b) expected form.

it is known that the sequence q_{mn} has a quicker convergence to the limit Q_0 as m and n tend to infinity. This process has been applied to the ratio β_0^{k+1}/β_0^k, and the results are given in *table II*.

Therefore, the radius of convergence is 1.2572 and the point $\varepsilon_0 = 1.2572$ is a singularity of the function solution. This value is the same for all β_n, up to five digits. For β_0, we can say that ten digits are valid because they do not change in the Richardson extrapolation.

3. Analysis of the solution singularity

Up to the point ε_0, the solution is analytic in the interval $]-\varepsilon_0, \varepsilon_0[$, and has a real singularity at the value ε_0. It would be interesting to know the exact nature of the singularity, and if this singularity corresponds to a bifurcation point. Let the aspect ratio $e(\varepsilon) = a(\varepsilon)/b(\varepsilon)$ be a characteristic quantity which qualifies the solution. Of course e has the same singularity, i.e. ε_0. It is possible to calculate the limited series defining $e(\varepsilon)$ by a division procedure. This function is plotted in *figure 6(a)*.

The graph of $e(\varepsilon)$ is not compatible with the expected singularity (26). The aspect ratio should be of the form given in *figure 6(b)*. To analyse the paradox, the inverse function $\varepsilon = \varepsilon(e-1)$ is considered. Its series is given by inverting the series $e(\varepsilon)$. One can see in *figure 7* that this function has a horizontal tangent for the values $(e-1 = -0.61206,\ \varepsilon = 1.2572)$. The maximum value of ε corresponds to the value of ε_0 up to three digits, which confirms the existence of a singular point, see Vainberg and Trenoguine [11].

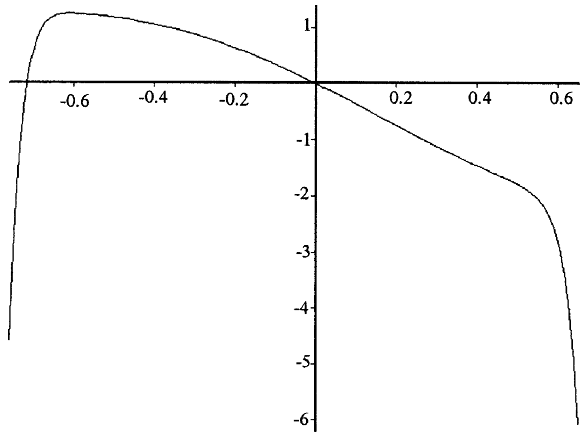

Figure 7. Plot of the function $\varepsilon = \varepsilon(e-1)$.

Table III. Values of γ_n for each β_n.

n	γ_n	n	γ_n
0	0.2056	6	0.1173
1	0.1976	7	0.1093
2	0.1778	8	0.0909
3	0.1591	9	0.0801
4	0.1373	10	0.0757
5	0.1296		

Returning to the general analysis of the construction of the solution given in Section 2.3, it can be seen, from the implicit function theorem that, as long as the operator $(\partial G/\partial u)(u(\varepsilon), \varepsilon)$ is not singular, there is an analytic solution. Therefore, when the solution is no longer analytic, the operator is no longer invertible, which is a condition for ε_0 to be a bifurcation point or a limit point, this last category contains the turning points. One expects a singularity of the type given in (26). There are several possibilities for evaluating γ. First with (27) and the value of ε_0, we can determine γ by interpolating the slope in the Sykes–Domb graph. The results obtained by this method are not reported. We preferred a second approach because of its precision: the logarithmic derivative method. For a function f defined as a series having a singularity as in (26), we can calculate the derivative $d/d\varepsilon \log f$ and determine the residue in the Laurent series of the Padé approximants $P[n, 1]$ of this derivative, see Baker [12], this residue is equal to γ.

For each β_n, i.e. each n, the values of γ are different but are always in the interval $]0, 1[$. The two methods give results that coincide up to two digits (*table III*).

The sequence γ_n seems to converge to zero as n tends to infinity, but there is no sufficient numerical evidence of this fact.

4. Conclusion

We have shown that the perturbation domain method is a correct method for obtaining the asymptotic expansion for free surface problems. In order to evaluate the domain of validity of this method, we have studied the shape of a bubble (or a drop of fluid) in the irrotationnal flow of a perfect fluid. We have obtained an explicit expression for the solution of this free boundary problem. It should be noticed that this solution is valid up to the singularity of the solution. A singularity is not necessarily a bifurcation point. It could be also a limit point. For this problem it would be a turning point. In order to decide whether the singularity is a bifurcation point or a turning point, the spectrum of the linear operator $(\partial F/\partial u)(u(\varepsilon), g(\varepsilon), \varepsilon)$ at ε_0 (see relation (9) for notations) has to be computed. Unfortunately we could obtain no more than seven terms for the different series needed, and this is not enough to be certain of the conclusion. For example, the zero of the first eigenvalue of this operator was not exactly ε_0, but 1.98. In either case (bifurcation or limit point) the branch of the solution we have exhibited still exists beyond the critical value ε_0, and we could characterise it. Miksis et al., see [13], have studied the shape of a deformed bubble numerically, in a uniform flow, a problem very similar to the one treated here. However they do not keep the volume constant. They observe similar shapes but for greater values of the Weber number. When comparing the shape obtained by these authors, we notice that they obtained more elongated shapes. They plotted (*figure 4(b)*) the volume as a function of the Weber number, which is twice our parameter ε. The volume varies from 0 to 11, i.e. over a decade. In this paper, the curve in *figure 5*, where the Weber number is plotted versus the aspect ratio ($b(\varepsilon)/a(\varepsilon)$ in the cited paper) has a horizontal tangent, which implies the existence of a turning point. They obtained a “maximum Weber number above which the solution fails to exist”. This maximum Weber number is 3.23. This value corresponds to $\varepsilon = 1.615$, a value which is close to the one we obtained. Meiron, in [14], studied exactly the same question as in this paper. He used a collocation method in spherical co-ordinates for small values of the Weber number. He noticed difficulty in the convergence of his calculations for a value of the aspect ratio $(b(\varepsilon)/a(\varepsilon))$ less than 1.4. In our calculations, the series is convergent up to the value of 2,5777 for the same aspect ratio. In *figure 2* of his paper, he plotted the shapes for $1.806 \leqslant 2W_e^2 \leqslant 3.0492$. In our notation, these values correspond to $0.903 \leqslant \varepsilon \leqslant 1.5246$. This latter value is greater than the singular value that we obtained. He never observes the pinching of the bubble. All we can say is that, in the common range of parameters the shapes are somewhat different.

We have tried the method on the problems of a rotating drop with interfacial tension and rotating fluid holding together by gravitation. The bifurcation points are completely determined for these two problems (see Appell [1]). The series solutions we obtained were valid up to the bifurcation points. So the perturbation method is powerful. However, it must be mentioned that the combinatorial blow-up leads to an exponential growth of the swap, used by the computer, with the number of terms of the series, as it is illustrated in *figure 8*.

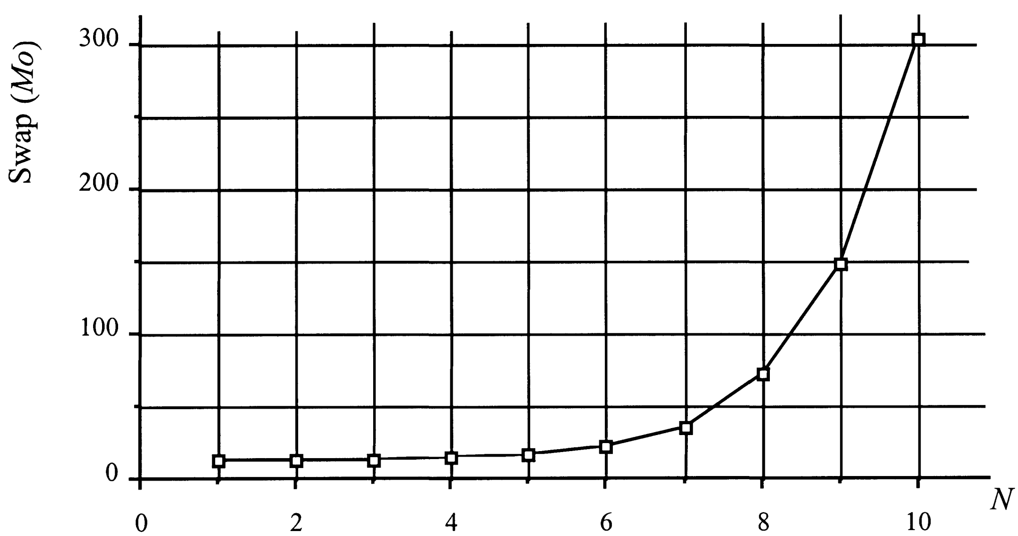

Figure 8. Swap used in Mo (megabytes), versus the number of terms in the perturbation series.

The calculation with a symbolic code must not be made as usually done 'by hand'. We used some tricks of computational algebra to accelerate the computation, but we observe the combinatorial blow-up anyway. This is a drastic limitation of any perturbation method.

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